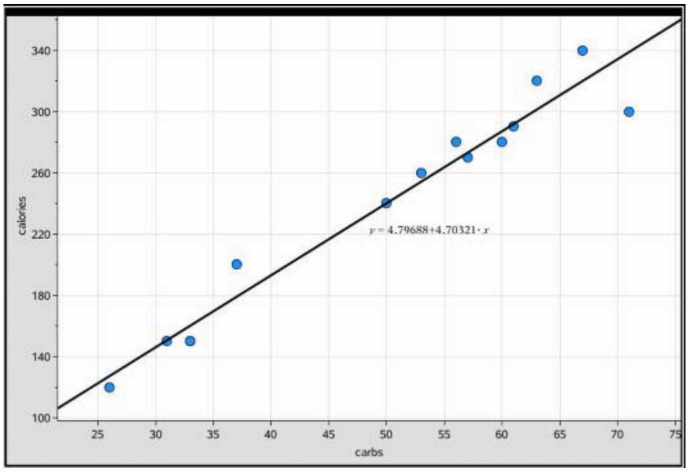
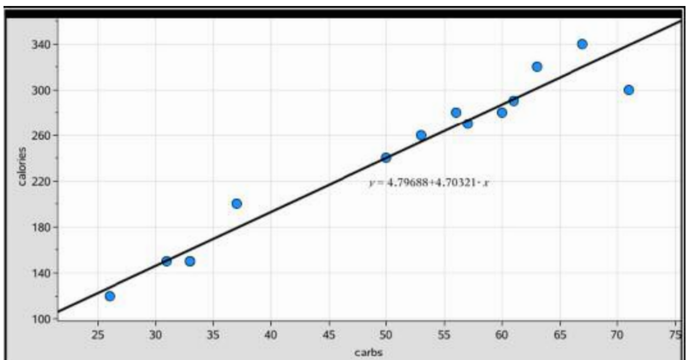


As an extension, it is interesting to note that every increase of one gram of carbohydrates contributes to an increase in 4.7 kCal. You can see this in the slope of the line. A little research determines that protein and carbohydrates contribute about 4.1 kCal/g, which is pretty close to this value. The y-intercept of approximately 4.8 is likely contributed by the fat in the drinks.



"a"	4.79688
"b"	4.70321
"r²"	0.952524
"r"	0.975974

The fit of this line is very strong. Looking at the results here and the graph, you can see that the correlation coefficient and the coefficient of determination are both in the high 90% range.

There is a similar, but less strong, relationship between fat and calories, and similarly between fat and carbohydrates. ▲

"No employment can be managed without arithmetic, no mechanical invention without geometry." Benjamin Franklin

▲ Hey, It's Elementary: Rich Tasks and Big Ideas... Then and Now

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Lynda Colgan's career has included roles as a classroom teacher, a university professor, and newspaper columnist. Her contributions to mathematics and its teaching have been recognized through awards such as the Marshall McLuhan Foundation Distinguished Teacher Award. Lynda always exhibits a passion for mathematics and views her professional mission as dispelling the myth that math is the bad guy.

I began a six-month sabbatical on January 1, 2016—a wonderful opportunity to go back through files and take stock of what is a "keeper" and what is best bound for the recycling heap. As I worked my way through drawers and boxes of articles, first drafts, half-written pieces, and one-page brainstorm doodles, I came across what I believe is a wonderfully rich Grades 7/8 task: *A Population Mystery*.

The task was developed in collaboration with my former Scarborough Board colleague, Peter Harrison, for our work in professional development with Intermediate teachers, in part, as our response to a presentation by Dr. Rafaella Borasi (now Dean of the Warner School of Education at The University of Rochester) at the 1997 Canadian Mathematics Education Study Group Conference at Lakehead University in Thunder Bay, Ontario.

At the presentation, Dr. Borasi emphasized the fact that "…professionals are rarely, if ever, presented with a well-defined problem and expected to apply known mathematical models to come up with an objective solution. Rather, the task is most often presented to them in the more vague and open-ended form of a 'problematic situation.'"

Dr. Borasi's comment resonated with Peter and myself, and over many long hours, we discussed the inauthentic nature of so many of the "real world" problems that Grades 7/8 students faced in class, and the shallowness of many of the conceptual demands that were placed on student expectations.

Our resolve to generate a suite of rich tasks was galvanized when we read Howard Gardner's 1998 letter to *The Harvard Education Review*. In his eloquent piece, Gardner described what he believed to be the mathematical skills, knowledge, and values that should be emphasized in schools:

> …adults need to be able… to make prudent decisions in light of the laws of probability;… individuals should understand the nature of scientific enterprise: what is a fact, a theory, a hypothesis, an experiment. Individuals should be able to understand a newspaper report about a new medical treatment, evaluate the evidence on which it is based, and decide whether to believe it.

Gardner went on in his letter to argue that our goal, as mathematics educators, should be to teach and measure the complex web of the "big ideas" of mathematics that are at the heart of mathematical thinking and informed decision making.

Gardner went on to describe "big ideas" as the connecting concepts and processes that go beyond content: mathematical reasoning, mathematical modelling, problem solving, and communication, using multiple representations. Gardner's definition of "big ideas" was the notion of the family of conceptual connections that empower students both to see mathematics as an integrated whole and to make thoughtful connections among related mathematical notions and purposeful applications to other domains.

In response to Drs. Borasi and Gardner, Peter and I set our sights on developing problematic situations for middle school students that reflected Gardner's words, i.e., "to give students a sense of what it means to understand a topic in detail, and one hopes, whet their appetites for continuing to learn and understand deeply for the rest of their lives."

This particular task was designed to show that learning tasks in the Data Management and Probability strand need not be limited to spinners, dice, playing cards, and bingo chips. Our goal was to challenge students to interpret the probability that surrounds us daily by illustrating the importance of justifying real-world conclusions, using mathematically defensible arguments.

After an introduction to the task, its "big ideas," and a description of the curriculum expectations for the task and context, what follows is a "teaching episode"—the task is presented as interactive discourse among teacher and students and students and students.

Overview of the Activity

In this problematic situation, students attempt to solve a population mystery based on statistical information about human populations and bear populations. They are given the data in the form of a set of graphs, and they are required to analyze the information, looking for clues that will help them to evaluate the validity of a researcher's conclusions about a bear population. To develop their argument, students must apply probabilistic arguments to interpolate and extrapolate values/decisions from the visual data, and use both experimental and theoretical probabilities in concert with other factors, to guide their conclusions.

Big Ideas

Based on Gardner's contention that graphing, statistics, and probability are extremely important domains because of the fact that we are living in an information age, we elected to turn to the various forms of print media to determine how information is categorized, summarized, and displayed. We presented our problematic situation in such a way that students would be required to apply and integrate skills and knowledge from graphing (displaying information) with those from statistics (discussing data numerically) and probability (the mathematics of outcomes).

With much large-scale assessment data suggesting that while students are able to make direct readings from graphs and tables, but unable to articulate relationships among data, we sought, through this problematic situation, to challenge students to examine graphical displays, using a qualitative lens, e.g., looking for symmetry and other patterns that can form and inform quantitative arguments.

To develop their argument, the students must assess the strengths, weaknesses, and biases of samples and data-collection methods; critique the ways in which the statistical information and conclusions are presented; and evaluate decisions based on probability that will require a combination of theoretical calculations, experimental results, and subjective judgments.

Prior Knowledge

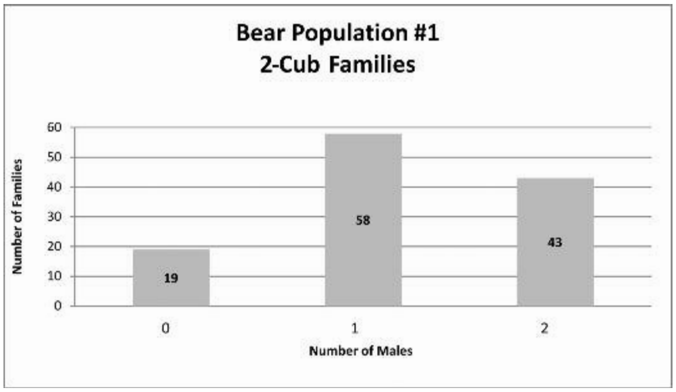
This problematic situation assumes that students will be familiar with the numerical definitions of probability (including conversions among decimal, percent, and fractional forms) and the strategies for listing all the possible outcomes for an event (i.e., probability paths, tree diagrams).

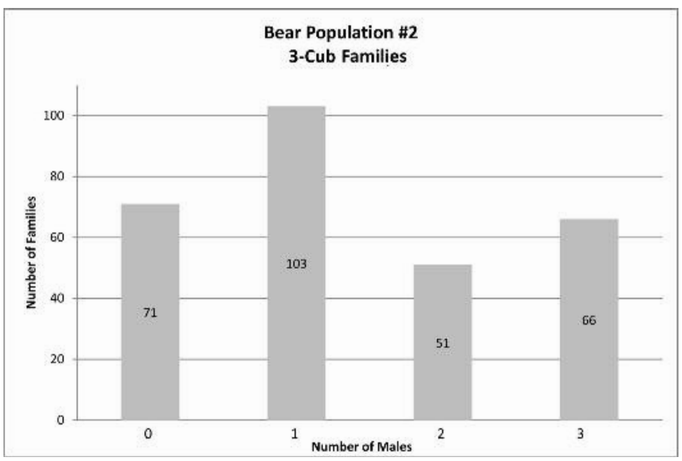
The Classroom Context

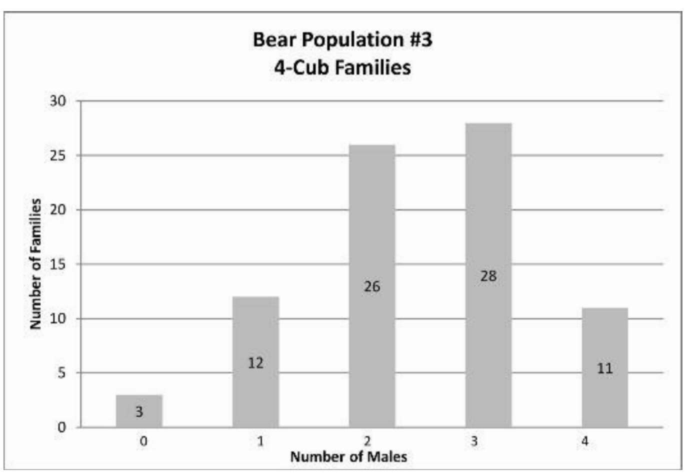
Our teacher, Ms. B., believes, as did Socrates, that good teaching occurs when one instructs, delights, and moves another. Her math classroom was energetic: one in which students regularly worked collaboratively. Ms. B. was enthusiastic about challenging her students with this genuine dilemma, and she and her students were happy to have us become temporary members of their classroom.

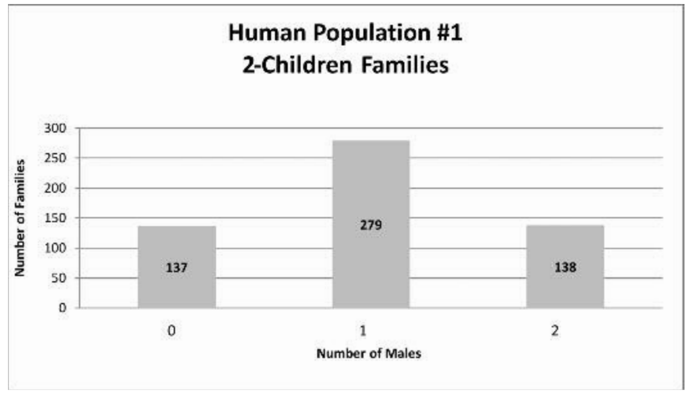
The Problematic Situation

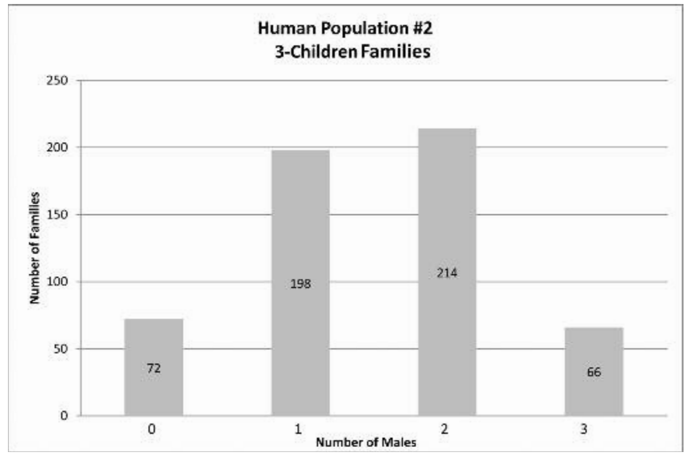
A team of researchers in Northern Ontario has been tracking bear families and keeping track of the number of male cubs in each family. They think that they have collected some rather odd findings. To help understand their data, they have also assembled some similar data for humans. They surveyed a series of human families with similar numbers of children and made a note of the number of boys. All of the results are shown in graphical form. The researchers concluded that the probability that a mother bear will give birth to a male cub is different from the probability that a human mother will give birth to a son.

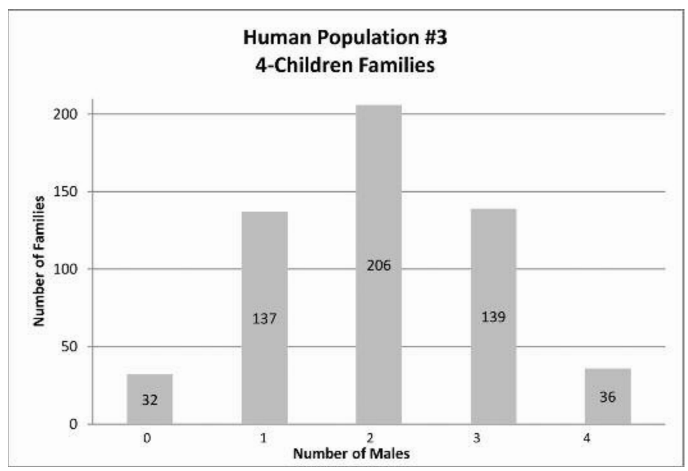












Ms. B. These graphs certainly contain a lot of information. I'd like someone to give us a quick summary of the story that these graphs tell.

Anne I'll try. We have six bar graphs. Each one is about either bear families or human families, and they only tell us about the number of male offspring—either kids or cubs—and they are paired off so that the bear and human families with the same number of offspring are next to each other.

Ms. B. Thanks, Ann. Why is it important for the graphs to be side by side?

Ian It makes it easier to compare. If we look at the second row, we have all the families with either 3 cubs or 3 kids. You can see right away that there were 15 bear families with no male cubs and 72 human families with no male kids.

Ms. B. Good point, Ian. But what if you look at the overall shapes of the graphs instead of focusing on the numbers? What stories do the graphs tell then?

Ian The human graphs are easier to describe. They're pretty balanced. The bear graphs are a little lopsided. I'm not sure what that means though.

When no one offers an answer immediately, Ms. B. suggests that they look at the graphs again and then discuss Jeff's point with the other students in the pod. After about five minutes, she calls the class's attention.

Ms. B. Well, what do you think?

Hoda I think that you'll always have the same number of boys and girls because it's all about chance. When a baby is born, it will either be a boy or a girl. It's a 50-50 chance thing.

Ms. B. Jeff, you started the conversation about "balance" in the graphs for humans. Who can add a little more description to the shape of the graph?

Mollie I think it is important that the graphs are symmetrical. There's a hump in the middle. The shorter bars are off to the end and the taller bars are in the middle.

Ms. B. How is this conversation like another probability conversation that we had in class recently?

Stefan The coin-toss experiment! When we tossed two coins, we found out that we got one head and one tail more often that either two heads or two tails. So, instead of heads and tails, we have boys and girls.

Ken Are you saying that the chance of tossing a head is the same as getting a boy in the family?

Ms. B. At this point, we need facts to support our ideas. Where can we find evidence to help us understand this situation?

Ann I think we need to make a tree diagram for families with two babies.

Ms. B. Great suggestion, Ann. In pairs, I'd like you to construct a tree diagram for Graph #2—human families with two children. Is there a pair who would be willing to draw their tree diagram on chart paper to help us in our discussion later? Thanks. Hoda and Jeff have volunteered.

Ms. B. gives the students about 12 minutes to work on the task and then calls for the class's attention as Hoda and Jeff display their tree diagram.

Ms. B. Jeff or Hoda, would either of you please walk us through your tree diagram?

	# OF MALES	0	1	2		
TWO CUBS	REPORTED	19	58	43		
	EXPECTED	19	58	43		
	# OF MALES	0	1	2	3	
THREE CUBS	REPORTED	15	71	103	51	
	EXPECTED	15	69	103	52	
	# OF MALES	0	1	2	3	4
FOUR CUBS	REPORTED	3	12	26	28	1 1
	EXPECTED	2	12	28	28	10

Jeff I'll start. This is the same as the coin-toss experiments. If you have one coin, there are two options: heads or tails. If there is one child, there are two possible outcomes: boy or girl. If there are two coins, you have four possibilities: two heads, two tails, or one head and one tail, and vice versa.

Hoda Then we remembered that we also used fractions with our tree diagrams for the coin-toss experiments, so we did the same thing here, and then we calculated percents too.

Jeff The graph tells us that there are 550 families in all. Just add up the numbers on the bars. 25% of the families should have zero boys. And they do—one-fourth of 550 is 137.5.

Stefan This experiment was fixed. The theoretical and experimental probabilities match perfectly.

Ms. B. Good point, Stefan. To test the scientists' conclusions, we'll need to create similar trees for 3- and 4-children families and check out the numbers.

Ms. B. asks the students to form groups of four. One pair in each group is given responsibility for the 3-children families, and the other pair constructs the tree diagram for the 4-children families. She asks them to check each other's work. After 15 minutes, she calls for the class's attention. There is general agreement that the numbers on the human graphs are very accurate reflections of what probability would predict. A lively discussion ensues about how "real" the data is because the students feel that the human families are "too perfect."

Ms. B. Great discussion. Thanks to everyone for participating. Now let's turn our attention to the graphs of the bear families.

Mollie Well, those graphs sure aren't perfect. The bear graphs all slant in favour of boys. If you look at the 3-cub families, there are 51 families with 3 males and only 15 families with 3 females.

Tony How do you know that, Mollie?

Mollie If you look at the graph, you can see that there are 15 3-cub families with zero male cubs, so that means the family must have all female cubs. And in the 71 3-cub families with 1 male cub, there must be 2 female cubs.

Ms. B. Thanks for clarifying, Mollie. I'm sure that there were others in addition to Tony who had the same question.

Ann There sure are more male bear cubs. I wonder why. Does that mean the male cubs are stronger and have a better chance of surviving in the wild? Or maybe it's not 50-50 for bears like it is for humans. Maybe it's 60-40 or 70-30.

Ms. B. We are missing a lot of information. All we have are the graphs and the information that they contain. We know the graphs are skewed toward male cubs. So, what are our next steps?

Hoda I think Ann is right. We can't use 50-50 chances for bears.

Jeff So you mean we have to use trial and error when we do the calculations and then see how close our experimental numbers are to the actual numbers?

Hoda Ms. B., is that right?

Ms. B. We won't know unless we try. I think the suggestion is good. We have to conjecture about our probability: 55%, 60%, 65%, 70%, and so on, and then compare our results with the data that we have in the graphs. So, for homework, Pod A will work with 40% and we'll continue along the alphabet to Pod H trying 75%. I'm looking forward to your findings. See you tomorrow.

The Next Class

Ms. B. We are going to take a few minutes so that everyone in each pod can compare their solutions with the others in your group. When your pod is satisfied that they have a solution that everyone can explain, we'll move into expert groups to compare solutions for all the probabilities from 55 to 75.

Ms. B. circulates around the classroom to be sure that each pod has a correct solution, and provides support and help as required. When she is confident that the expert groups have come to consensus about the best possible solution, she asks for a volunteer to present to the class. Ken, Stefan, and Mollie come to the board and record their table.

Ms. B. Thanks to our three volunteers. Who is going to report?

Mollie I'll start. I started by making a tree diagram. When I created the tree for a 4-cub family, I got 16 possibilities, like FFFF, MFFF, MFMM, and MMFM. I noticed that four of my arrangements had 3 males and 1 female. I tried working with 40% probability for a male and started doing the calculations. <insert chart>

Ken We weren't really sure what to do when we had our numbers. But then Stefan turned back to the coin-toss experiments from last week and we remembered that we had to multiply the probabilities together.

Stefan To get the expected number of 3 males, we multiplied 0.4 x 0.4 x 0.4 x 0.6 x 4 x 80 and got 12.3. Too small.

Mollie So if 40% was too small, we knew it had to be another number. The group that tried 60% was almost perfect. They multiplied 0.6 x 0.6 x 0.6 x 0.4 x 6 x 80 to get 27.6.

Ms. B. Are there any questions or comments?

Eugene I didn't do the problem that way. I looked at the graph for 3-cub families, and calculated the total number of cubs and the total number of males. There are 15 families with no males, so 0. There are 71 families with 1 male, so that's 71 male cubs. There are 103 families with 2 males, so that's 206 more male cubs. Last, there are 51 families with three cubs; that's 153 male cubs. So in all, there are 0 + 71 + 206 + 153, or 430, male cubs. Then I found out how many cubs in all. That's 240 families with 3 cubs each, or 720. The probability that a cub is male is 430/720, which is about 60%. I tried it for all the graphs, and my method works.

Ms. B. So we have two different approaches that lead us to the same solution. So, what should we say to the researchers about their conclusion?

Hoda We can tell them that their graphs do tell us that the probability of a mother bear cub is 60%.

Ann But I'd like to tell them that their graphs don't tell the whole story without knowing information about the number of cubs born in the litters, compared to the number of cubs in the family when the tally was taken. Maybe more newborn female cubs die because of poor health or a predator.

Ms. B. Interesting. That leads right into your final task, which is to write a report for the researchers, critiquing the presentation of their results and their conclusion, and suggesting ways to improve the accuracy of their study and the media reports about their study.

Peter and I hope that you found this dialogic format, based on classroom field tests of one of our attempts to formulate a rich task, from nearly 20 years ago, to remain a highly illuminative example of the role of “rich learning tasks” and “big ideas” in teaching and learning mathematics today.

And now I’ll return to the next box of files. Who knows if another treasure awaits? ▲

Encyclopedia of Mathematics Education (S. Lerman, 2014) Excerpts

“Over the past several decades, changes in perspective as to what constitute statistics and how statistics should be taught have occurred, which resulted in new content, pedagogy and technology, and extension of teaching to school level. At the same time, statistics education has emerged as a distinct discipline with its own research base, professional publications, and conferences. There seems to be a large measure of agreement on what content to emphasize in statistics education: exploring data (describing patterns and departures from patterns), sampling and experimentation (planning and conducting a study), anticipating patterns (exploring random phenomena using probability and simulation), and statistical inference (estimating population parameters and testing hypotheses)... The main sources of students’ difficulties were identified as: facing statistical ideas and rules that are complex, difficult, and/or counterintuitive, difficulty with the underlying mathematics, the context in many statistical problems may mislead the students, and being uncomfortable with the messiness of data, the different possible interpretations based on different assumptions, and the extensive use of writing and communication skills.”

Ben-Zvi, D. (2014). Data handling and statistics teaching and learning. In S. Lerman (Ed.), *Encyclopedia of mathematics education* (pp. 137–140). London, UK: Springer.

▲ What’s the Problem? Nifty Number Nine

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Welcome back, problem solvers. Last time, I left you with the following problem:

Pick any whole number between 20 and 100, inclusive. Add the digits together and subtract the sum from the original number. Finally, add the two digits from your last result and concentrate really hard.

You have probably seen this problem, or a variation of it, at some time or another. This problem came from *The Magic of Math* by Arthur Benjamin. Arthur Benjamin is a mathematics professor at Harvey Mudd College in California. He has an entertaining TED Talk video, where he performs mental arithmetic that is worth checking out.

This is a great one to give to the kids, have them run through it, and see their surprise when you announce that their number is 9. You or your students may have seen similar “number-predicting tricks” that can be explained with algebra. A common theme is shown in Figure 1.

Step	Example	Algebra
Pick a number	17	x
Add 13	17 + 13 = 30	$x + 13$
Double the result	30 x 2 = 60	$2(x + 13) = 2x + 26$
Subtract 8	60 – 8 = 52	$2x + 26 - 8 = 2x + 18$
Divide result by 2	52 ÷ 2 = 26	$(2x + 18) \div 2 = x + 9$
Subtract the original number	26 – 17 = 9	$x + 9 - x = 9$

Figure 1

Unfortunately, for the problem we are considering, we are combining a number with its digits, and the final result is the sum of the digits of our calculation. If x is the number, how do we deal with its digits? The spark of insight needed is that we don’t use a variable to represent our chosen number; rather, we use two variables to represent the digits. If we use T and U to represent the tens and units digits, respectively, then our original number is $10T + U$ and the sum of the digits